Reduced order modeling and system identification of nonlinear dynamical systems

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Data-driven modeling for nonlinear dynamical systems

From data $x$ find $f$ s.t.

$$\dot{x} = f(x)$$

**ISSUES:**
1) Dimensionality of the data
2) Correct coordinate system
3) Model identification
1) Dimensionality of data

• Data from simulations

Simulation of a flow around a cylinder by discretization of Navier-Stokes equations considering 73.131 degrees of freedom.

• Data from experiments

Video of millions of pixels

Two degrees of freedom!
2) Coordinate system

- **Nonlinear Burgers’ equation**

\[
 u_t + uu_x - \varepsilon u_{xx} = 0
\]

\[
 u = -2\varepsilon \frac{v_x}{v}
\]

\[
 v_t = \varepsilon v_{xx}
\]

**Geocentric**

**Heliocentric**

Linear heat equation

3) Model identification

- **Very few and interpretable** terms.
- **Nonlinearities** are typically **polynomials**

### Linear

\[ \nu_t = \epsilon \nu_{xx} \]

### Quadratic nonlinearities

\[ \omega_t + (u \cdot \nabla)\omega = \frac{1}{Re} \nabla^2 \omega \]

### Quadratic nonlinearities

\[
\begin{align*}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) - y \\
\dot{z} &= xy - \beta z.
\end{align*}
\]

Strategy

From $\mathbf{X}$, $\dot{\mathbf{X}}$ find $\mathbf{f}$ s.t. $\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$

**ISSUES:**
1) Dimensionality of the data
2) Correct coordinate system
3) Model identification

How to find $\phi$, $\psi$ and $\hat{f}$?

**ISSUES:**
1) Dimensionality of the data
2) Correct coordinate system
3) Model identification

- Dimensionality reduction
- Change of coordinates
- Identification of the dynamical model
- Restore dimensionality
- Return to original coordinates

Physical Dynamics

Reconstructed Physical Dynamics

- $x_1$
- $x_2$
- $\ldots$
- $x_N t$

- $\hat{x}_1$
- $\hat{x}_2$
- $\ldots$
- $\hat{x}_N t$

$t$
Strategy

Singular value decomposition
\[ X = U\Sigma V^* \]

Retain first \( d \) modes: \( X \approx \tilde{U}\tilde{\Sigma}\tilde{V}^* \)
Orthonormal set of coordinates

- Dimensionality reduction
- Change of coordinates

\[ \begin{align*}
  \tilde{U}^* & \quad \quad \quad \tilde{U} \\
  X & \quad \quad \quad Z \\
  \tilde{Z} = AZ & \quad \quad \quad \hat{X}
\end{align*} \]

Linear projection
Linear model
Nonlinear reduction
Nonlinear model

How to find \( \varphi, \psi \) and \( \hat{f} \) ?
From \( \mathbf{Z}, \dot{\mathbf{Z}} \) find \( \mathbf{f} \) which best fits the data, i.e.
\[
\dot{\mathbf{Z}} = \mathbf{f}(\mathbf{Z})
\]

\textbf{SINDy} provides a \textbf{nonlinear model} that can be still solved by \textbf{linear regression}:
\[
\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) = \xi_0 + \xi_1 z + \xi_2 z^2 + \cdots + \xi_m z^m + \cdots = \Theta(z) \cdot \xi
\]

where \( \mathbf{Z}, \dot{\mathbf{Z}} \) are given, \( \Theta \) consists in a set of candidate features and \( \Xi \) are the unknown coefficients. For instance:
\[
\Theta(\mathbf{Z}) = \begin{bmatrix} 1 & \mathbf{Z} & \mathbf{Z}^2 & \mathbf{Z}^3 & \cdots \end{bmatrix}
\]
\[
\Xi = [\xi_1 | \xi_2 | \cdots | \xi_n]
\]

Strategy

\[ \mathbf{X} \xrightarrow{\text{Linear projection}} \mathbf{\tilde{U}}^* \xrightarrow{\text{Nonlinear reduction}} \mathbf{Z} \xrightarrow{\text{SINDY}} \mathbf{\tilde{U}} \]

- How to perform **nonlinear** reduction?
- How to perform *simultaneously* dimensionality reduction, change of coordinates and system identification?
Autoencoder with SINDy

\[ \mathbf{X} \rightarrow \varphi (\mathbf{X}; \mathbf{W}_\varphi) = \mathbf{Z} \]

\[ \hat{\mathbf{X}} \rightarrow \psi (\mathbf{Z}; \mathbf{W}_\psi) = \hat{\mathbf{X}} \]

\[ \dot{\mathbf{Z}} \approx \Theta (\mathbf{Z}) \mathbf{E} \]

SINDY

**Loss function:**

\[ \| \mathbf{X} - \hat{\mathbf{X}} \|^2_2 + \lambda_1 \| \dot{\mathbf{Z}} - \Theta (\mathbf{Z}) \mathbf{E} \|^2_2 + \lambda_2 \| \mathbf{E} \|_1 \]

- **Autoencoder**
- **Latent dynamics**
- **Sparsity**

Minimize to find \( \mathbf{W}_\varphi, \mathbf{W}_\psi, \mathbf{E} \)

\( \rightarrow \varphi, \psi \) and \( \hat{f} \)
Forecast future trajectories

- Forecast physical behavior for unseen times
- Smart set of coordinates
- Parsimonious and interpretable model
- Speed up in computations
- Dimensionality reduction

**SINDy model**

\[
\dot{z} = \Theta(z, \beta) \Xi \\
z(t_0) = z_0
\]
Detect instabilities and bifurcations: continuation

- Track unstable branches and detect bifurcations
Application: Beam structure

Parameters:
- 28 forcing frequencies $\omega \in [0.526, 0.564]$,
- 2 forcing amplitudes $F \in \{0.25, 0.5\}$

Training set:
- 56 time-series of displacement $u$ and velocity $v$.

\[ \Theta(z, F \cos \omega t) \Xi \approx \ddot{z} \]

From 7.821 dofs to just 1
Beam – testing \textit{(online)}

\begin{itemize}
  \item \textbf{Encoding}
  \begin{align*}
    X(0) \xrightarrow{F, \omega} \varphi \\
    \hat{z}(0) \xrightarrow{f(z; F \cos \omega t)} \hat{z}_{\text{int}} \xrightarrow{} \hat{X}
  \end{align*}

  \item \textbf{Integration of low dimensional system}

  \item \textbf{Decoding}
  \end{itemize}

\textbf{Initial condition}
\textbf{Forcing values}

\textbf{Integration of low dimensional system}
\textbf{Decoding}

\textbf{Beam – testing (online)}
• Capability to track **unstable FRC branches** in the absence of data.
• **Generalization** for parameter instances $\omega, F$ out of the training distribution.

Fluid flow around a cylinder

The fluid exhibits different behaviors depending on the **Reynolds number**, that is related to the **inlet velocity**.

\[ z_1 = f_1(z; Re) \]
\[ z_2 = f_2(z; Re) \]
\[ z_3 = f_3(z; Re) \]

**Full order model**

\[ \frac{\partial \mathbf{v}}{\partial t} - \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla \cdot \sigma(\mathbf{v}, p) = 0 \quad (x, t) \in \Omega \times (0, T) \]
\[ \nabla \cdot \mathbf{v} = 0 \quad (x, t) \in \Omega \times (0, T) \]

**Latent model**

- **73.131** degrees of freedom.
- Computational time \((T = 30)\): \(\sim 40\) minutes.

- **3** degrees of freedom.
- Computational time: < 10 seconds.
Fluid flow – Testing (online)

- **Encoding**
  - Initial condition $X(0)$
  - Reynolds parameter $Re$

- **Integration of low dimensional system**
  - $\hat{z}(0)$
  - SINDy integration $f(z; Re)$

- **Decoding**
  - $\hat{z}_{\text{int}}$
  - $\psi$
  - $\hat{X}$

Initial condition: $X(0)
Re

Integration of low dimensional system:
- $\hat{z}(0)$
- SINDy integration $f(z; Re)$

Decoding:
- $\hat{z}_{\text{int}}$
- $\psi$
- $\hat{X}$

Fluid flow – Testing (online)
Results: $Re = 58$ (unsteady flow)

- FOM solution ($\sim 40$ mins): velocity magnitude

- AE+SINDY prediction ($< 10$ seconds): velocity magnitude
Extract dynamics insights with continuation

Latent system

\[ \dot{z}_1 = f_1(z; Re) \]
\[ \dot{z}_2 = f_2(z; Re) \]
\[ \dot{z}_3 = f_3(z; Re) \]

Latent periodic solutions

Physical periodic solutions

The proposed AE+SINDy approach allows to *simultaneously* achieve:

- **Dimensionality reduction**,  
- **Identification** of right set of **variables**,  
- **Identification** of an interpretable **dynamical model**,  
- Detection of the main dynamical features of the phenomenon such as **instabilities**, **bifurcations**, etc.

Possible directions for improvement:

- Improve system identification to deal with **noisy data** and inaccurate derivatives  
  → Ensemble SINDy\(^1\), weak-formulations\(^2\), ....
- **Normal form** identification of the observed dynamical system.

Thank you

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References: