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Reduced order modeling and system identification of nonlinear dynamical systems

Paolo Conti

Polytechnic University of Milan: **G. Gobat, S. Fresca, A. Manzoni, A. Frangi**.
AI Institute in Dynamic Systems: **N. Kutz, S. Brunton**.

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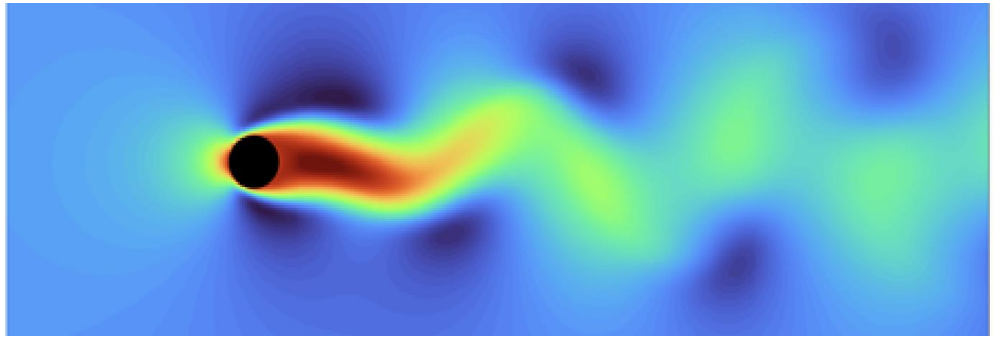
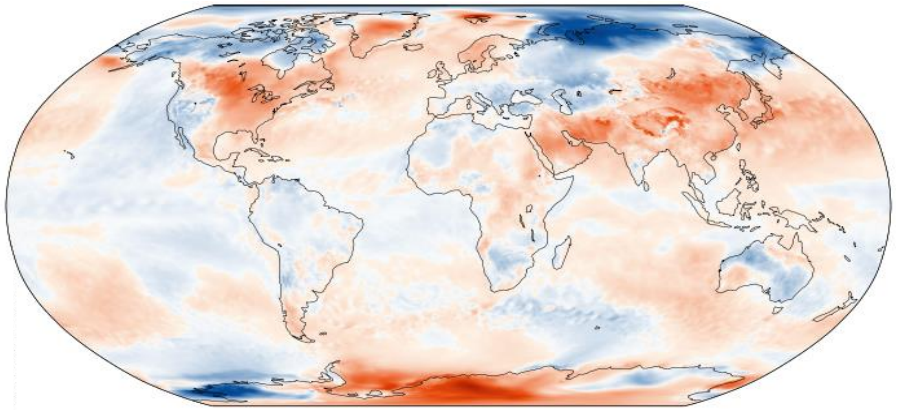
San Francisco, March 28

Data-driven modeling for nonlinear dynamical systems





Model

From data $\mathbf{x}$ find $\mathbf{f}$ s.t.

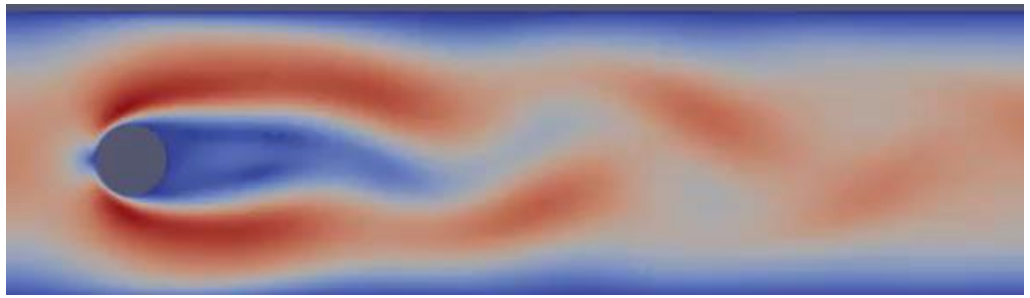
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

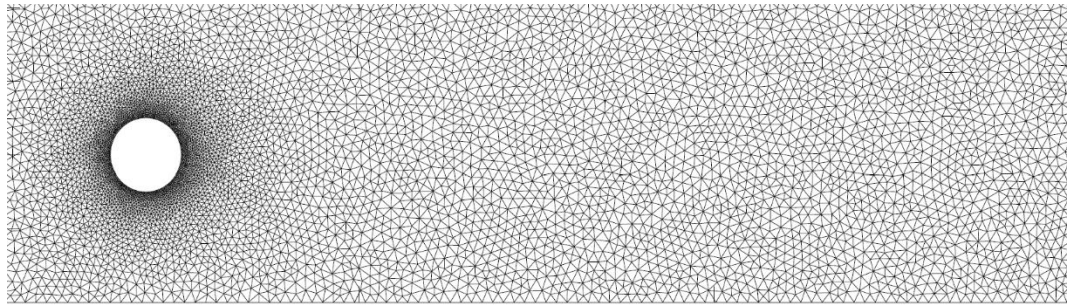
ISSUES:

1) **Dimensionality of the data**
2) **Correct coordinate system**
3) **Model identification**

1) Dimensionality of data

- **Data from simulations**

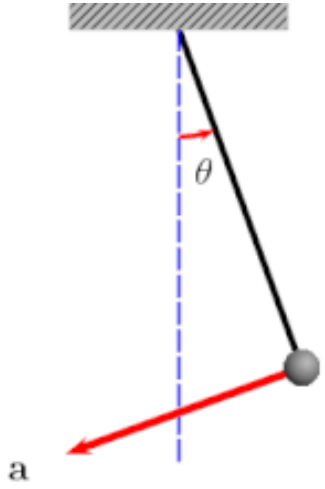




Simulation of a flow around a cylinder by discretization of Navier-Stokes equations considering **73.131 degrees of freedom**.

- **Data from experiments**

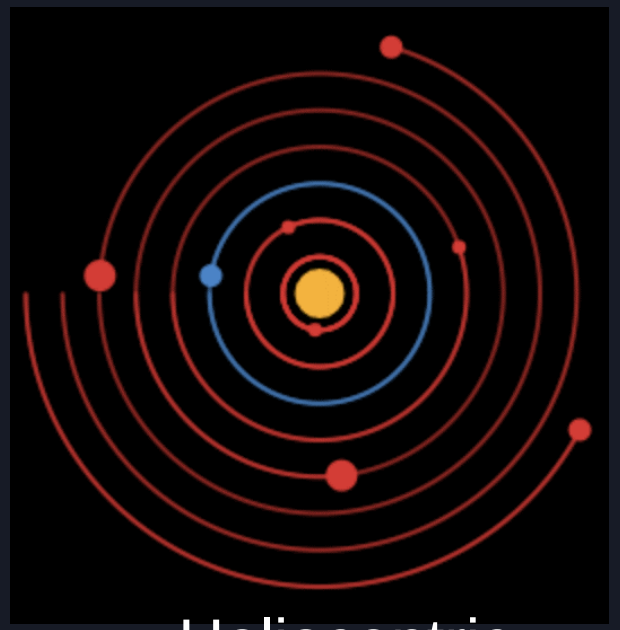
Video of milions of pixels



Two degrees of freedom!

2) Coordinate system

Geocentric



Heliocentric

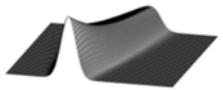
- **Nonlinear Burgers' equation**

$$u_t + uu_x - \epsilon u_{xx} = 0 \xrightarrow{u = -2\epsilon \frac{v_x}{v}} v_t = \epsilon v_{xx}$$

Linear heat equation

Nathan Kutz, J., Proctor, J.L. and Brunton, S.L., 2018. Applied Koopman theory for partial differential equations and data-driven modeling of spatio-temporal systems. Complexity, 2018, pp.1-16.

3) Model identification

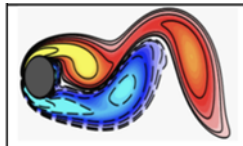


Burgers

$$u_t + uu_x - \epsilon u_{xx} = 0$$

Linear →

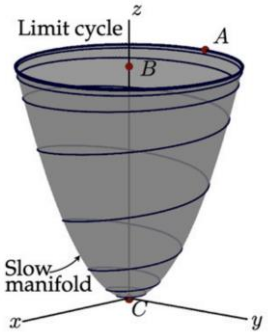
$$v_t = \epsilon v_{xx}$$



Navier Stokes

$$\omega_t + (\mathbf{u} \cdot \nabla)\omega = \frac{1}{Re}\nabla^2\omega$$

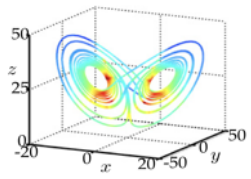
Quadratic nonlinearities →



$$\dot{x} = \mu x - \omega y + Axz,$$

$$\dot{y} = \omega x + \mu y + Ayz,$$

$$\dot{z} = -\lambda(z - x^2 - y^2).$$



Lorenz

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z. \end{aligned}$$

Quadratic nonlinearities

- **Very few** and **interpretable** terms.
- **Nonlinearities** are typically **polynomials**

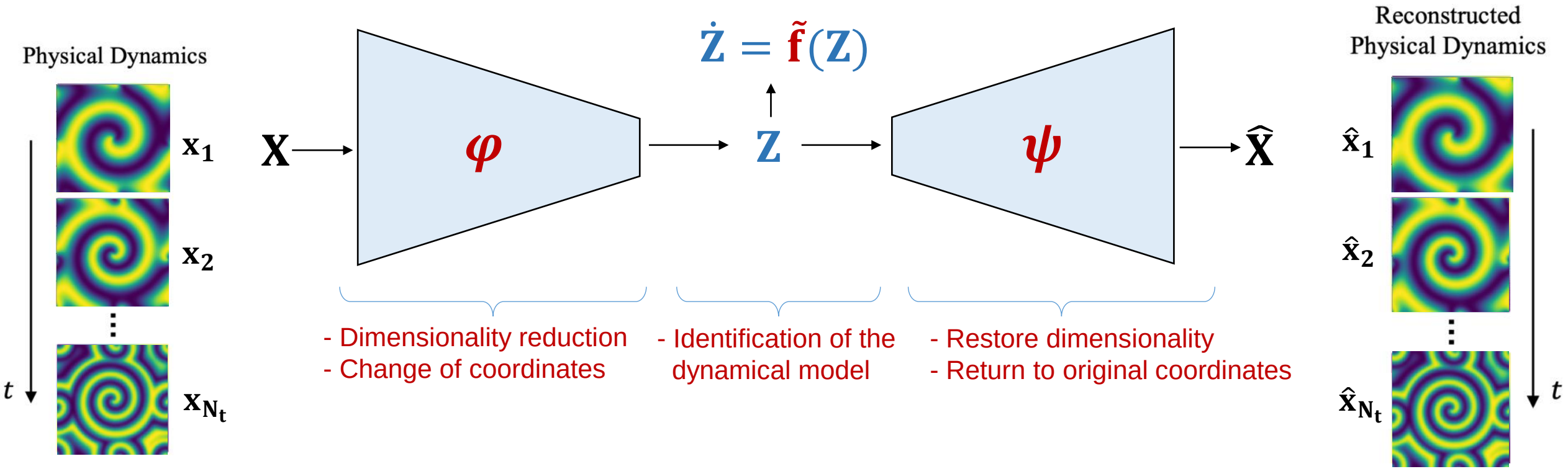
Rudy, S.H., Brunton, S.L., Proctor, J.L. and Kutz, J.N., 2017. Data-driven discovery of partial differential equations. Science advances, 3(4), p.e1602614.

Strategy

From $\mathbf{X}, \dot{\mathbf{X}}$ find $\mathbf{f}$ s.t.
$$\dot{\mathbf{X}} = \mathbf{f}(\mathbf{X})$$

ISSUES:
1) Dimensionality of the data
2) Correct coordinate system
3) Model identification

How to find $\boldsymbol{\varphi}, \boldsymbol{\psi}$ and $\tilde{\mathbf{f}}$?



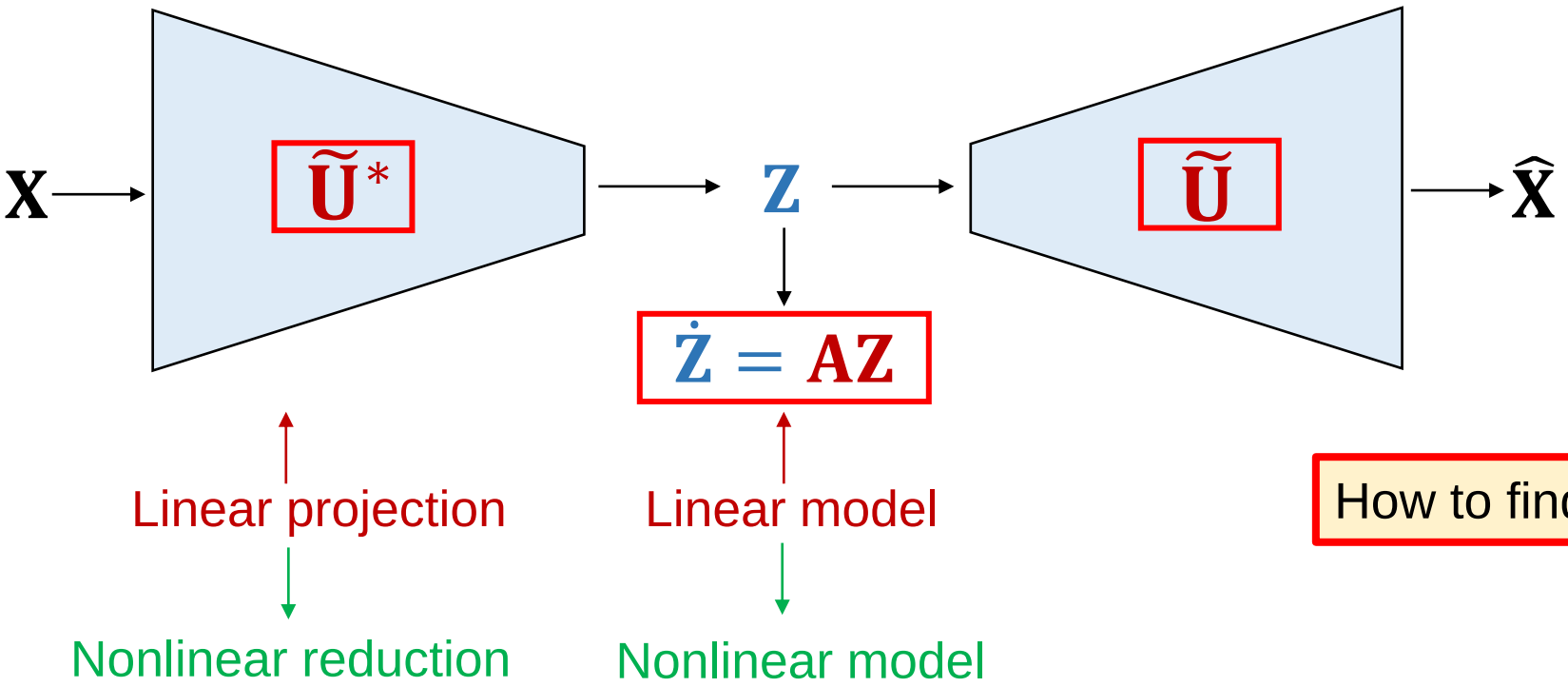
Strategy

Singular value decomposition

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$$

→

- Retain first d modes: $\mathbf{X} \approx \tilde{\mathbf{U}}\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{V}}^*$ — Dimensionality reduction
- Orthonormal set of coordinates — Change of coordinates



How to find $\boldsymbol{\varphi}, \boldsymbol{\psi}$ and $\tilde{\mathbf{f}}$?

Nonlinear modeling with SINDy(Sparse Identification of Nonlinear Dynamics) [1]

From $\mathbf{Z}, \dot{\mathbf{Z}}$ find $\mathbf{f}$ which best fits the data, i.e.

$$\dot{\mathbf{Z}} = \mathbf{f}(\mathbf{Z})$$

SINDy provides a **nonlinear model** that can be still solved by **linear regression**:

$$\dot{\mathbf{Z}} = \mathbf{f}(\mathbf{Z}) \approx \boldsymbol{\Theta}(\mathbf{Z})\boldsymbol{\Xi}$$

$$\dot{z} = f(z) = \xi_0 + \xi_1 z + \xi_2 z^2 + \cdots + \xi_m z^m + \cdots = \boldsymbol{\theta}(z) \cdot \boldsymbol{\xi}$$

where $\mathbf{Z}, \dot{\mathbf{Z}}$ are given, $\boldsymbol{\Theta}$ consists in a set of candidate features and $\boldsymbol{\Xi}$ are the unknown coefficients. For instance:

$$\underbrace{\Theta(\mathbf{Z}) = \begin{bmatrix} | & | & | & | & \\ 1 & \mathbf{Z} & \mathbf{Z}^2 & \mathbf{Z}^3 & \cdots \\ | & | & | & | & \end{bmatrix}}_{\text{KNOWN}}$$

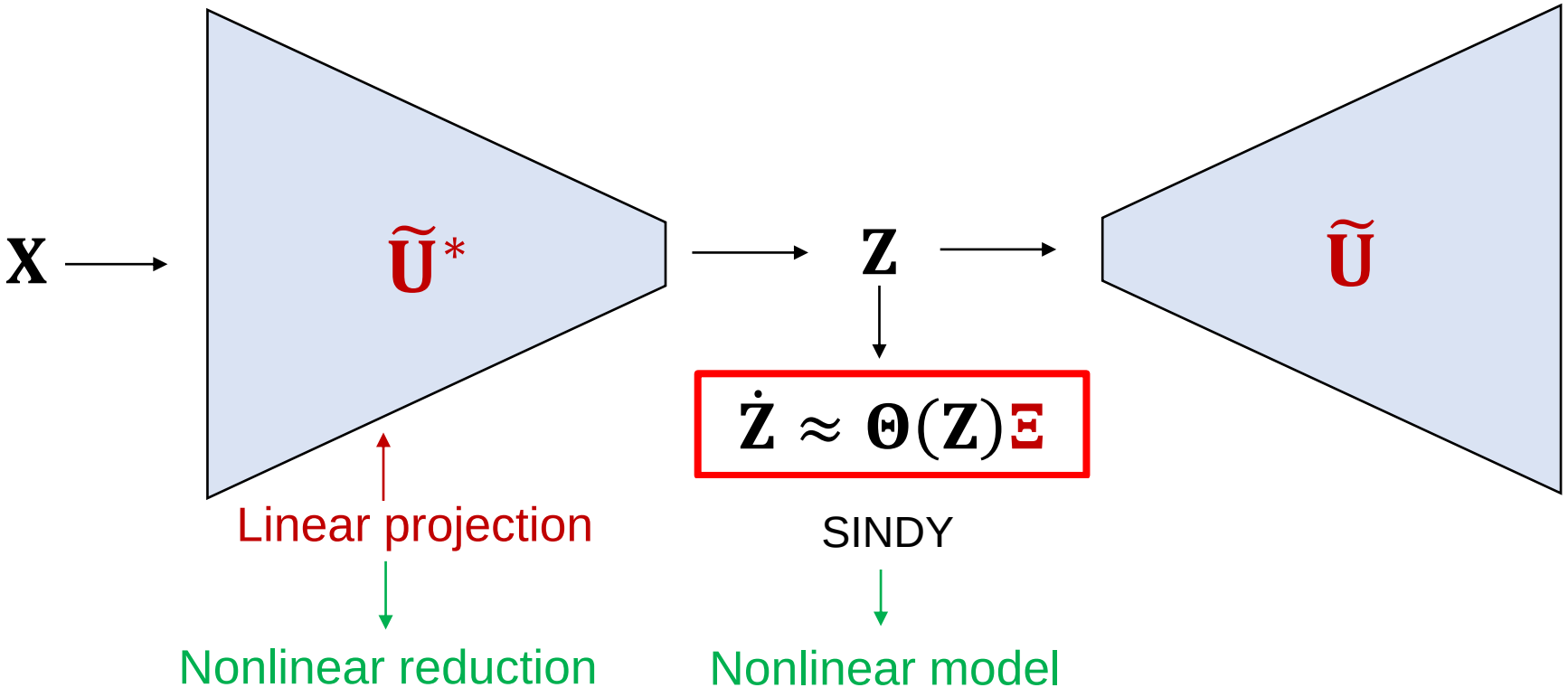
POLYNOMIALS

$$\underbrace{\boldsymbol{\Xi} = [\boldsymbol{\xi}_1 | \boldsymbol{\xi}_2 | \ldots | \boldsymbol{\xi}_n]}_{\text{UNKNOWN}}$$

SPARSE

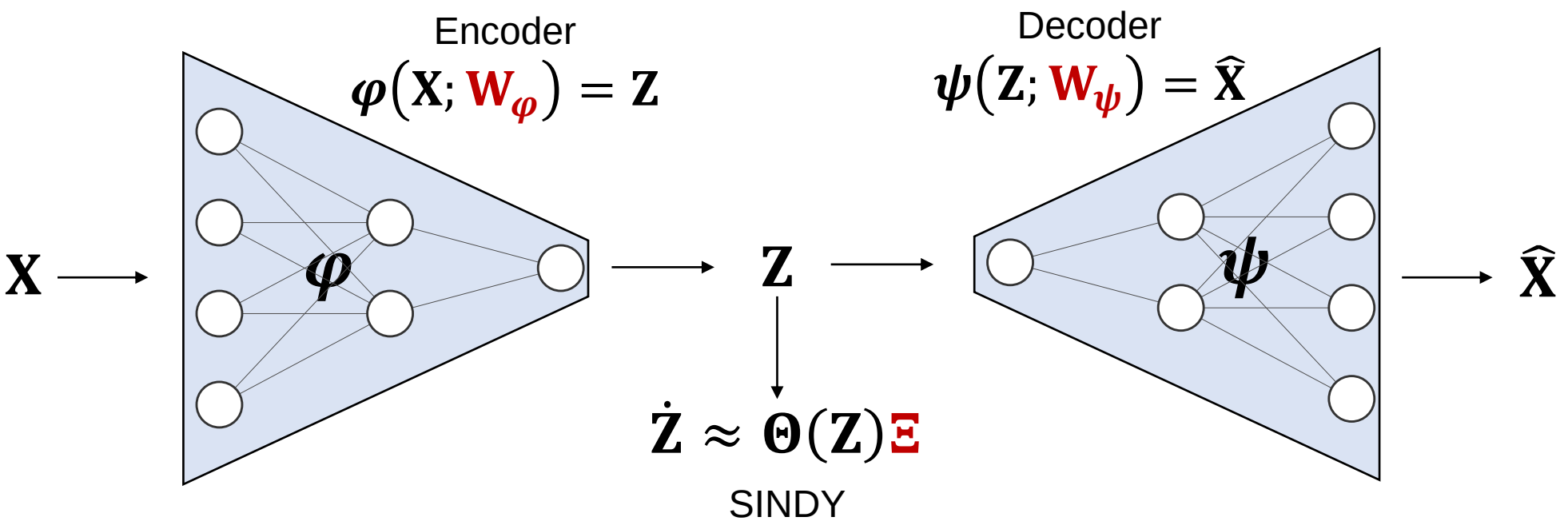
[1] Brunton SL, Proctor JL, Kutz JN. Discovering governing equations from data by sparse identification of nonlinear dynamical systems. Proceedings of the national academy of sciences. 2016 Apr 12;113(15):3932-7.

Strategy



- How to perform **nonlinear** reduction?
- How to perform *simultaneously* dimensionality reduction, change of coordinates and system identification?

Autoencoder with SINDy



Loss function:

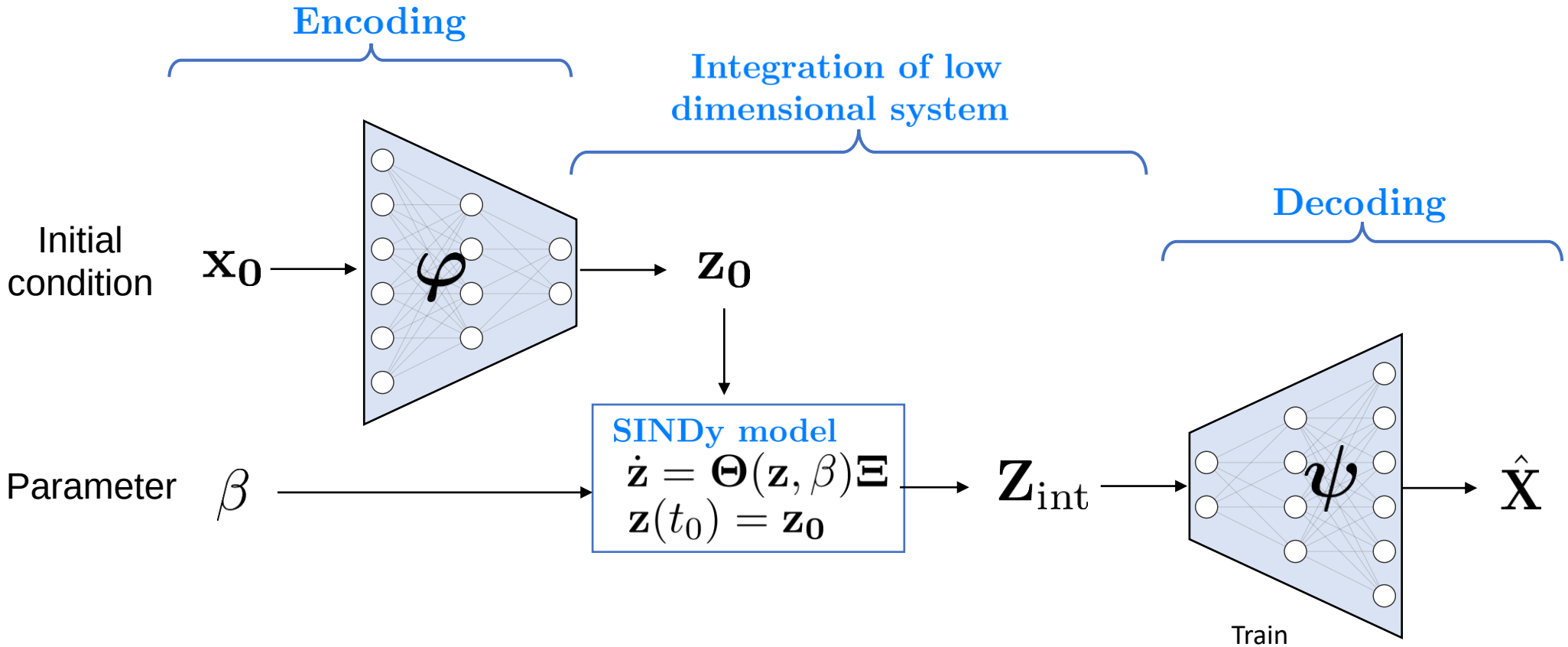
$$\underbrace{\|\mathbf{X} - \hat{\mathbf{X}}\|_2^2}_{\text{Autoencoder}} + \underbrace{\lambda_1 \|\dot{\mathbf{Z}} - \Theta(\mathbf{Z})\Xi\|_2^2}_{\text{Latent dynamics}} + \underbrace{\lambda_2 \|\Xi\|_1}_{\text{Sparsity}}$$

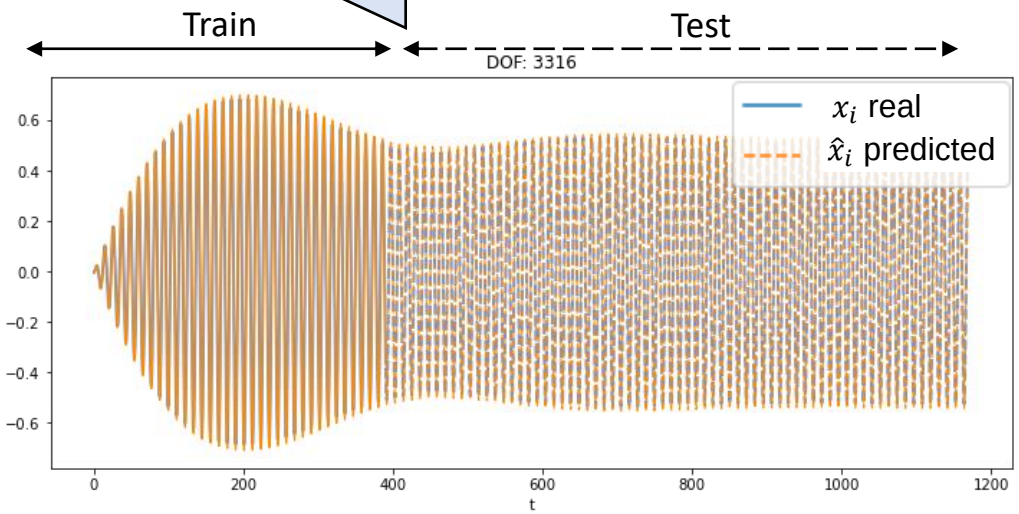
Minimize to find $\mathbf{W}_\varphi, \mathbf{W}_\psi, \Xi$

→ φ, ψ and $\tilde{\mathbf{f}}$

Champion K, Lusch B, Kutz JN, Brunton SL. Data-driven discovery of coordinates and governing equations. Proceedings of the National Academy of Sciences. 2019 Nov 5;116(45):22445-51.

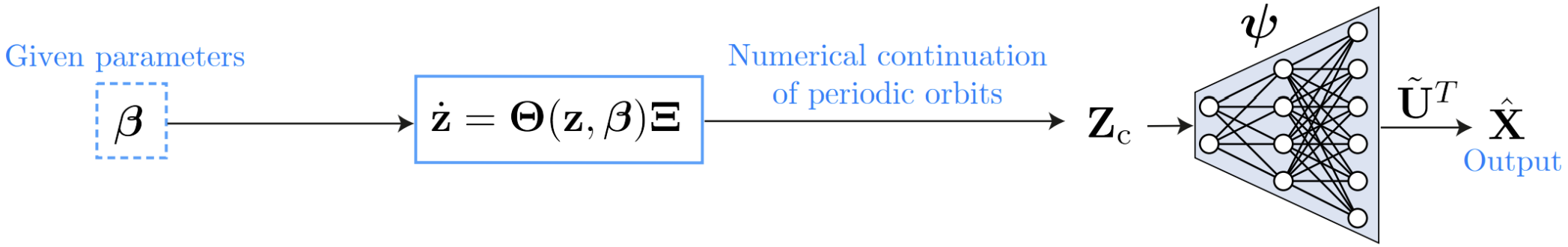
Forecast future trajectories

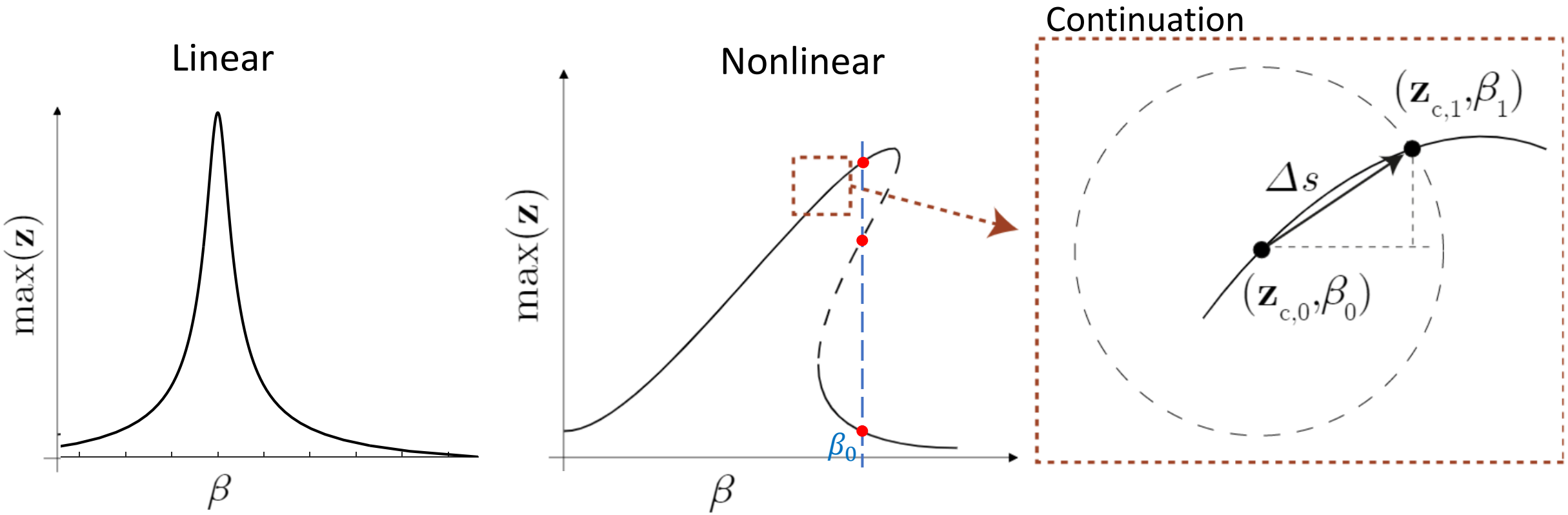




- **Forecast physical behavior for unseen times**
- **Smart set of coordinates**
- **Parsimonious** and **interpretable model**
- **Speed up** in computations
- **Dimensionality reduction**

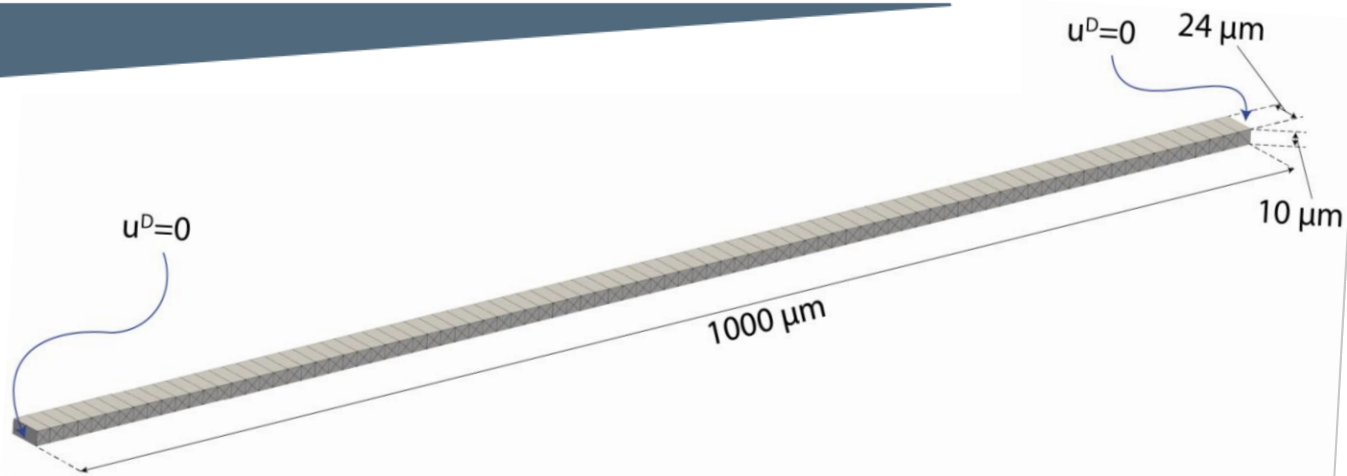
Detect instabilities and bifurcations: continuation

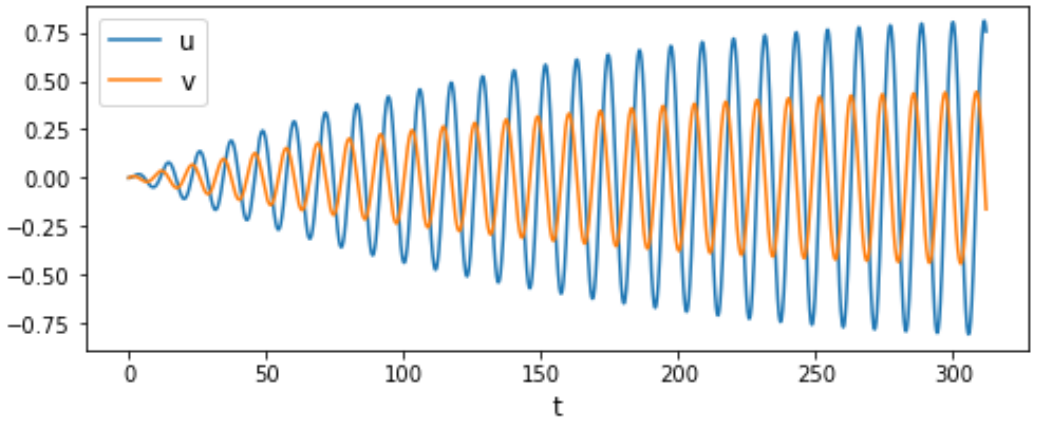




- **Track unstable branches** and **detect bifurcations**

Application: Beam structure





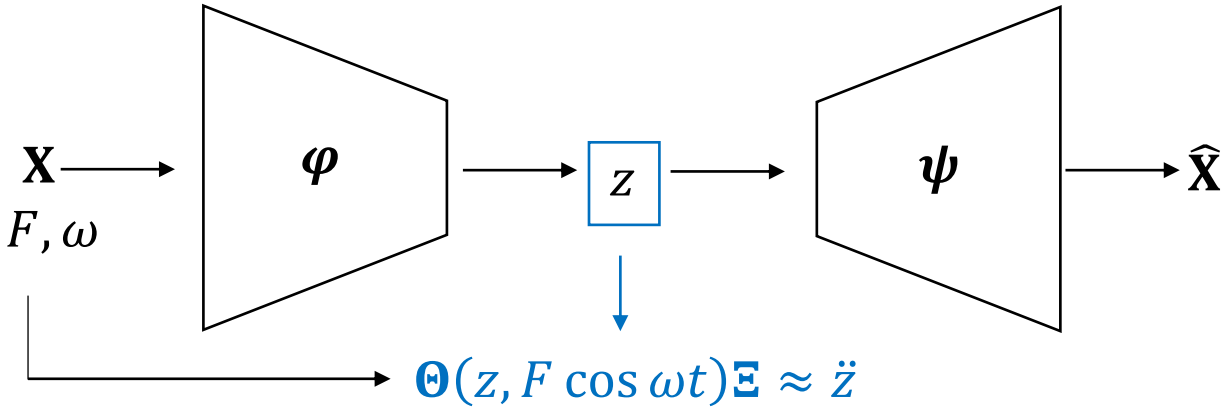
Parameters:

28 **forcing frequencies** $\omega \in [0.526, 0.564]$,

2 **forcing amplitudes** $F \in \{0.25, 0.5\}$

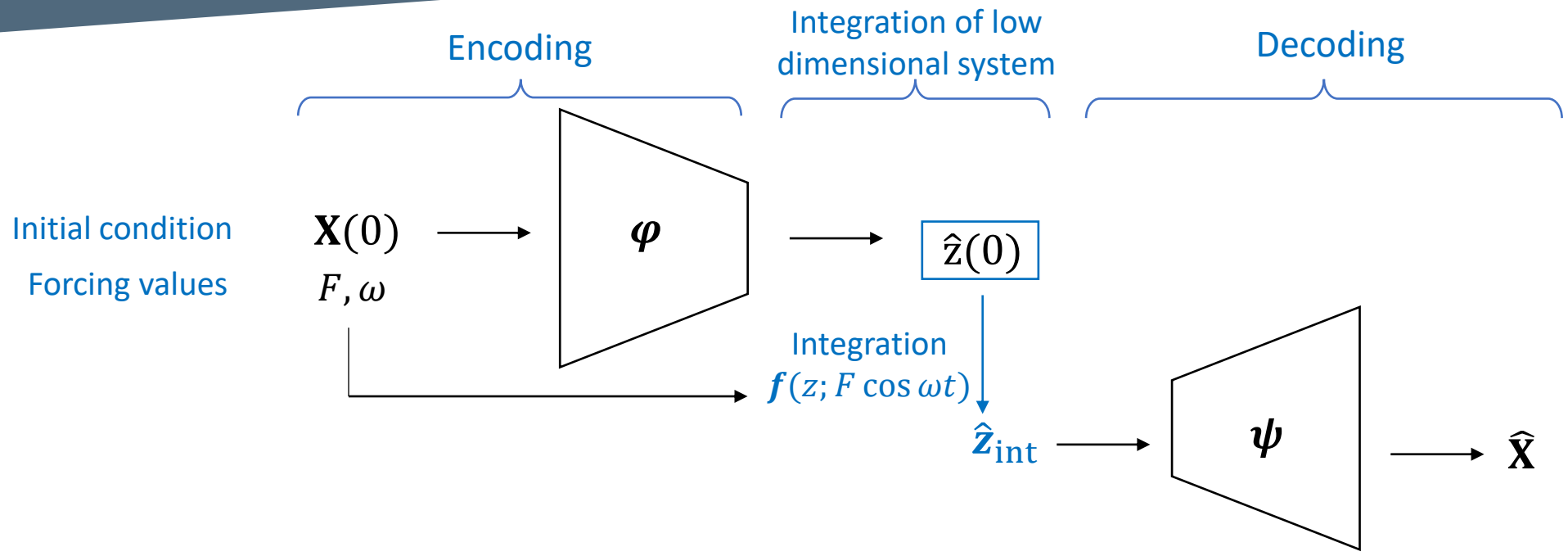
Training set:

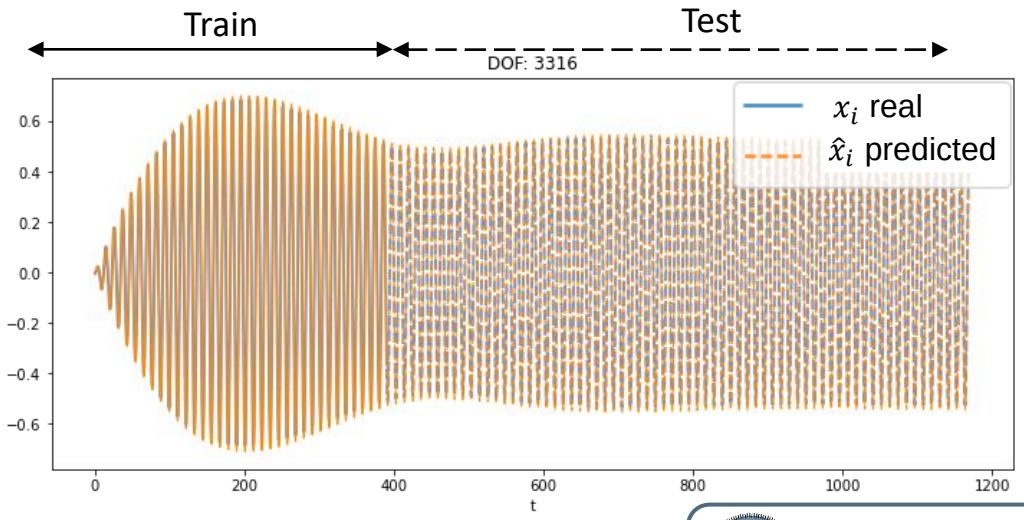
56 time-series of displacement *u* and velocity *v*.



From **7.821 dofs** to just **1**

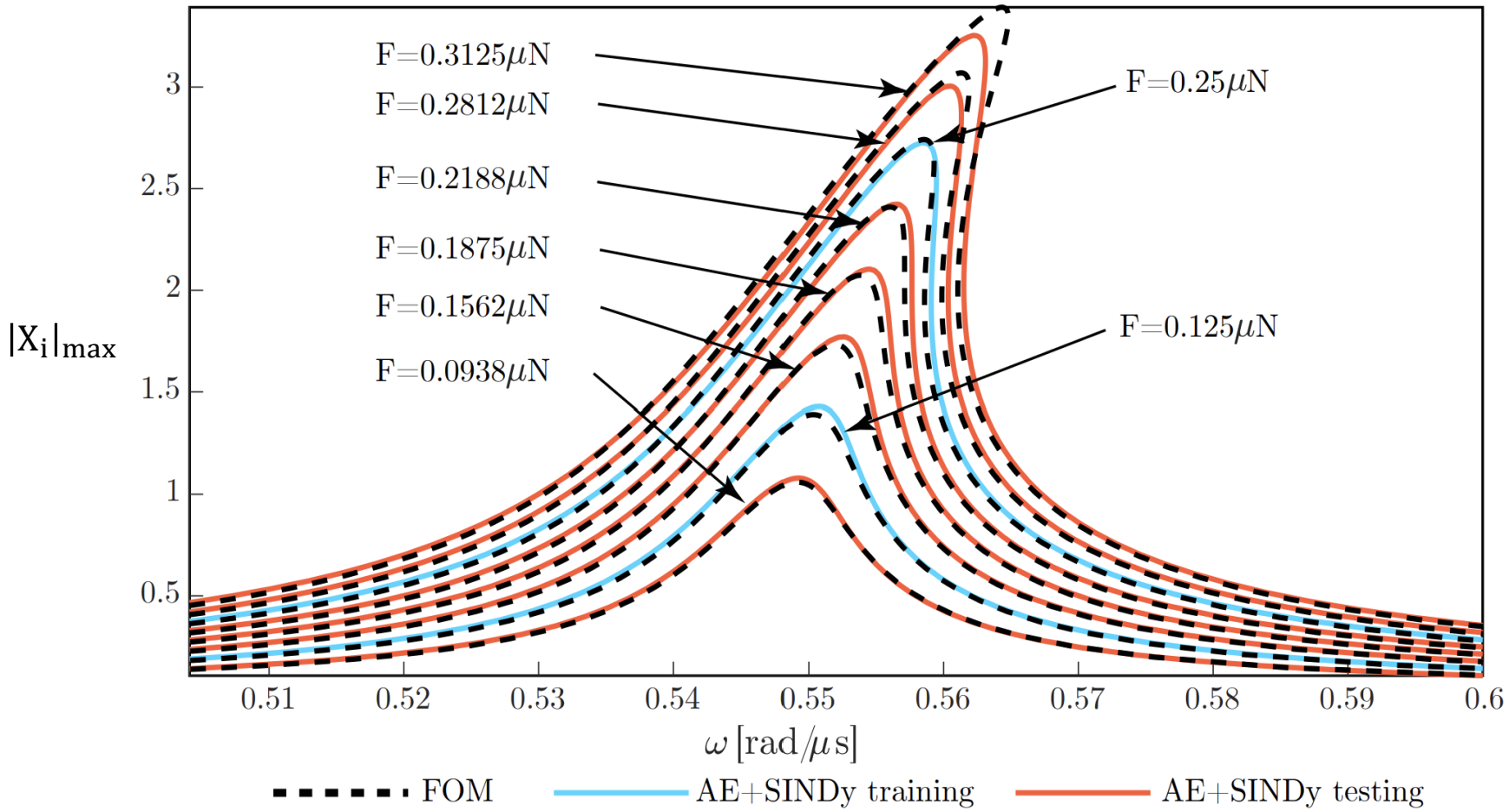
Beam – testing (*online*)





BEAM – continuation

Frequency response curves (FRC)

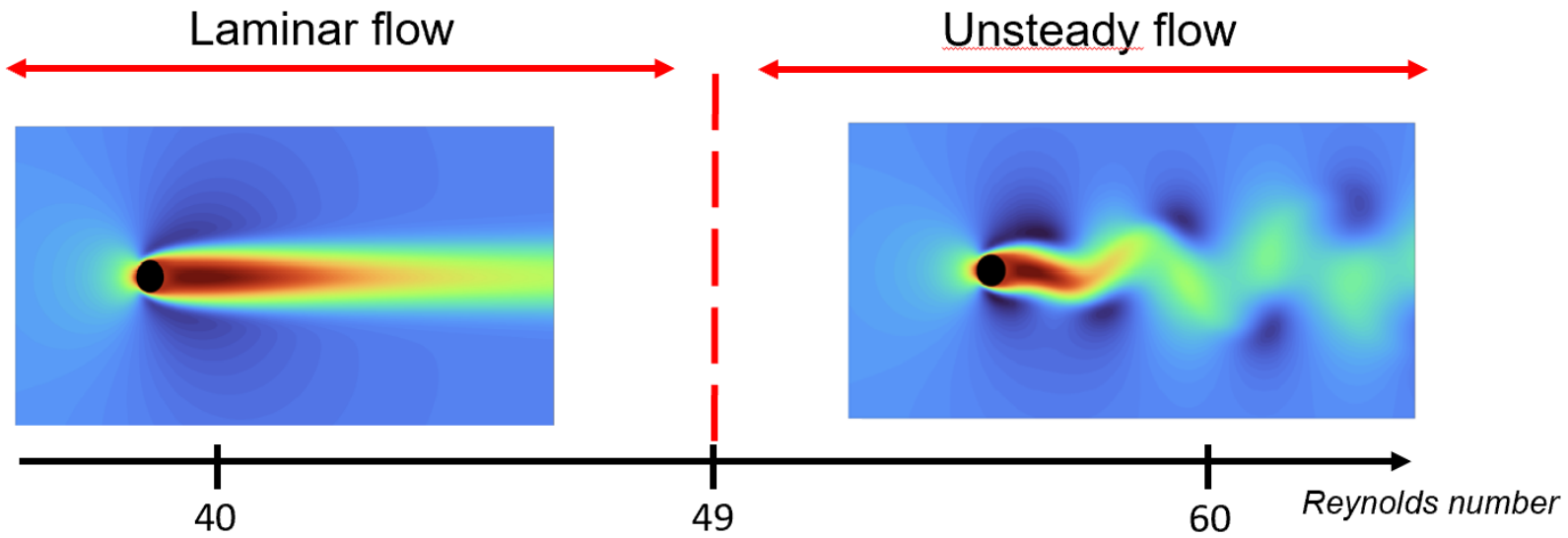


- Capability to track **unstable FRC branches** in the absence of data.
- **Generalization** for parameter instances ω, F out of the training distribution.

Conti, P., Gobat, G., Fresca, S., Manzoni, A. and Frangi, A., 2022. Reduced order modeling of parametrized systems through autoencoders and SINDy approach: continuation of periodic solutions. arXiv preprint arXiv:2211.06786.

Fluid flow around a cylinder

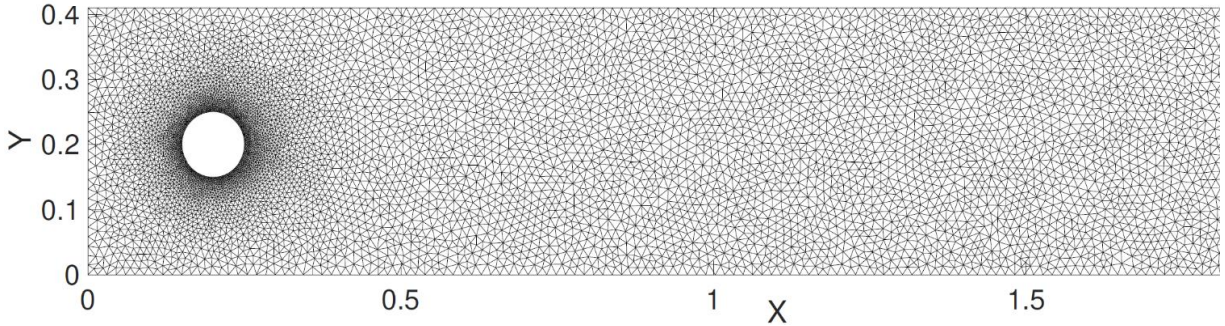
The fluid exhibits different behaviors depending on the **Reynolds number**, that is related to the **inlet velocity**.



Full order model

$$\rho \frac{\partial \mathbf{v}}{\partial t} - \rho \mathbf{v} \cdot \nabla \mathbf{v} - \nabla \cdot \boldsymbol{\sigma}(\mathbf{v}, p) = \mathbf{0} \qquad (\mathbf{x}, t) \in \Omega \times (0, T)$$

$$\nabla \cdot \mathbf{v} = 0 \qquad (\mathbf{x}, t) \in \Omega \times (0, T)$$



- **73.131** degrees of freedom.
- Computational time ($T = 30$): ~ **40 minutes**.

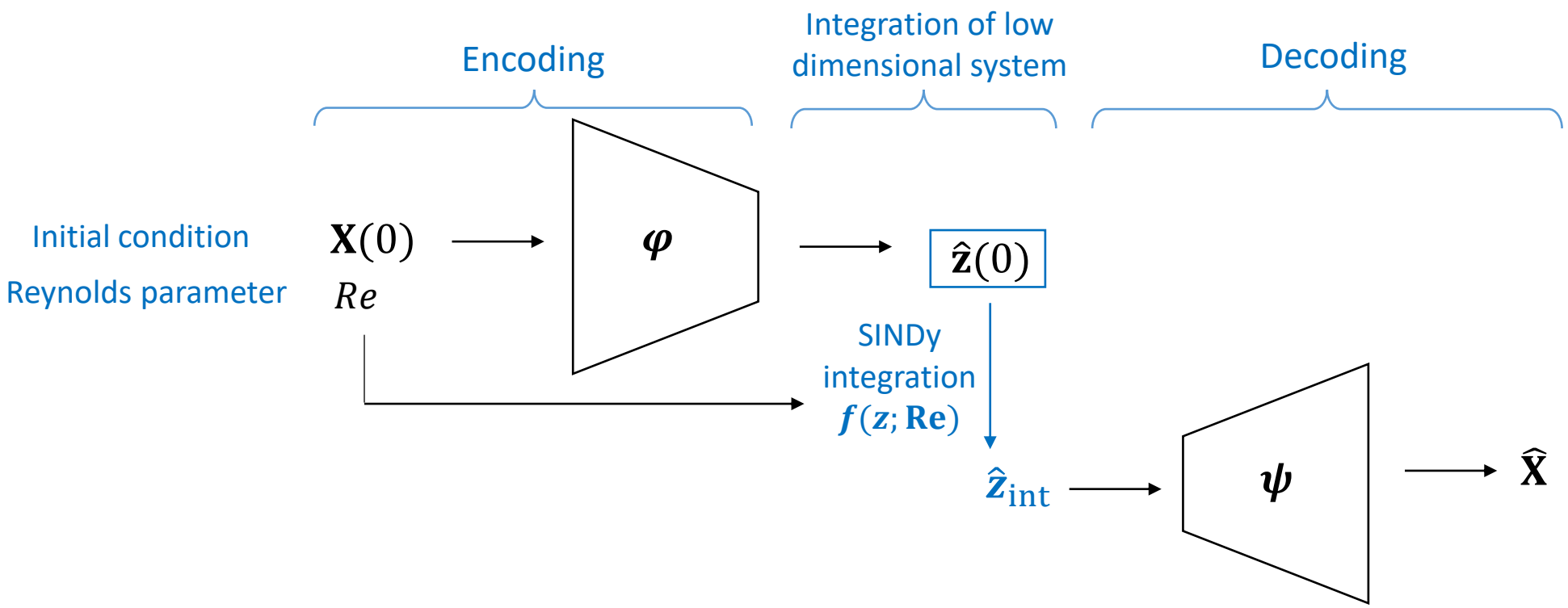
AE+SINDy

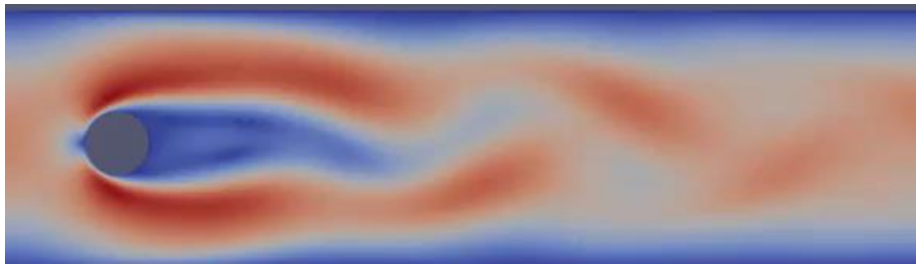
Latent model

$$\dot{z}_1 = f_1(\mathbf{z}; Re)$$
$$\dot{z}_2 = f_2(\mathbf{z}; Re)$$
$$\dot{z}_3 = f_3(\mathbf{z}; Re)$$

- **3** degrees of freedom.
- Computational time: $<$ **10 seconds**.

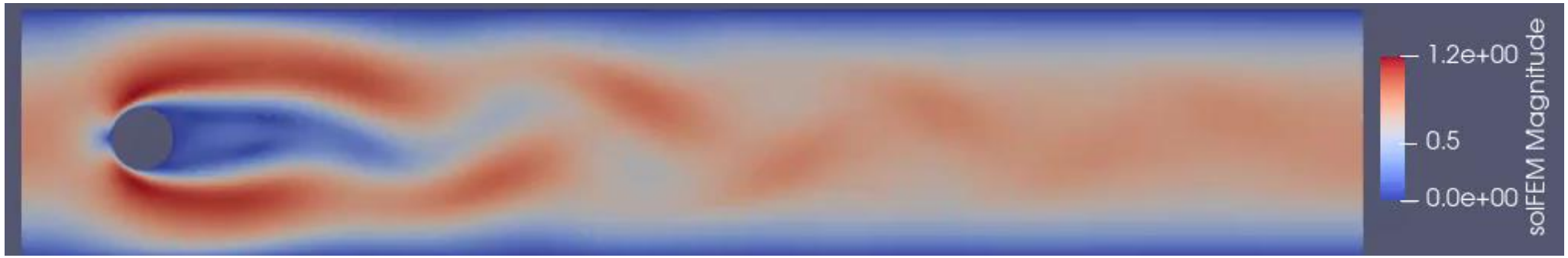
Fluid flow – Testing (*online*)



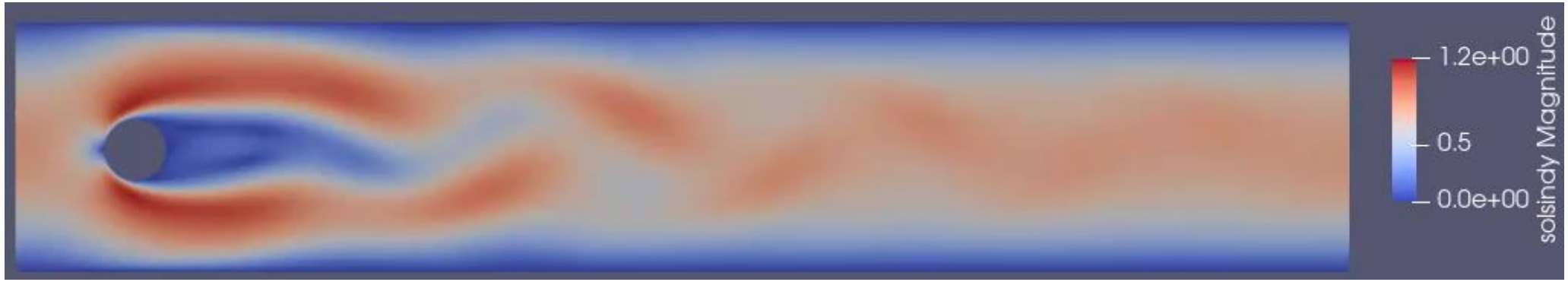


Results: $Re = 58$ (unsteady flow)

- FOM solution (~ 40 mins): velocity magnitude



- AE+SINDY prediction (< 10 seconds): velocity magnitude

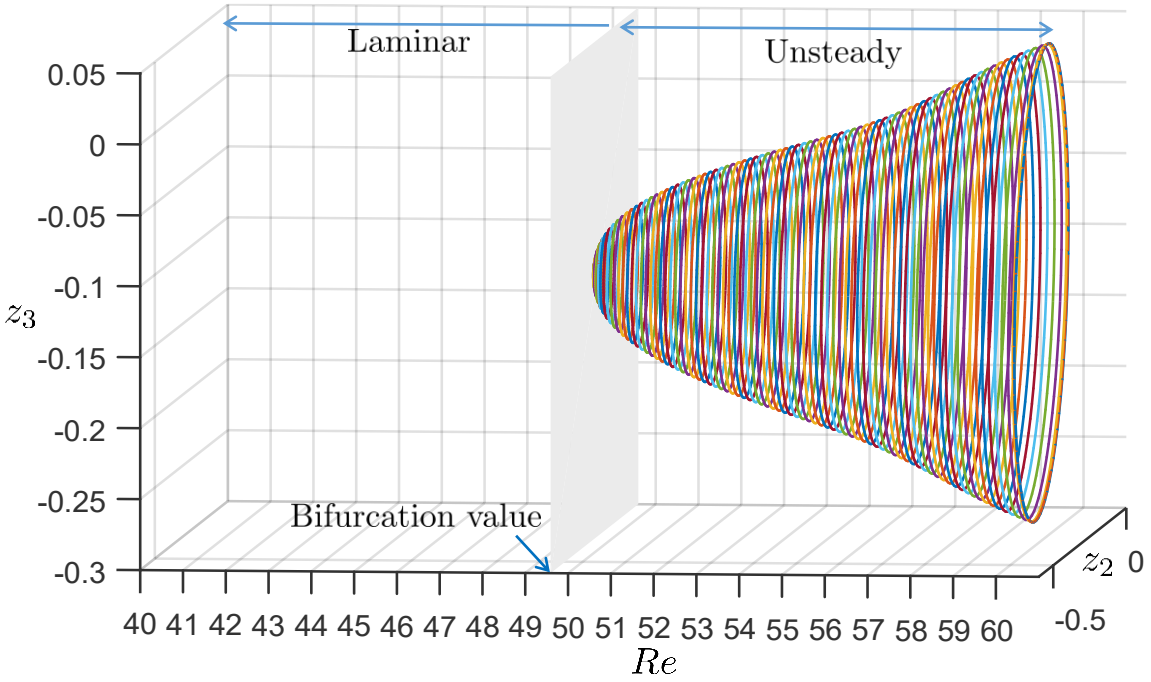


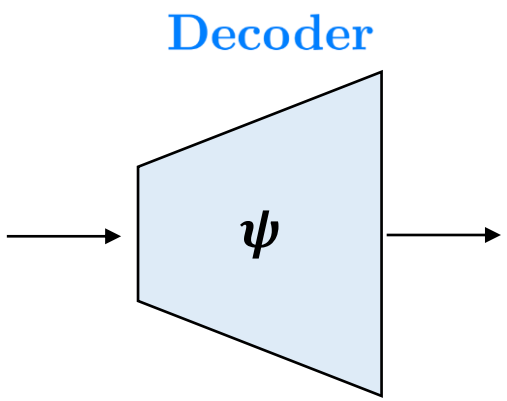
Extract dynamics insights with **continuation**

Latent system

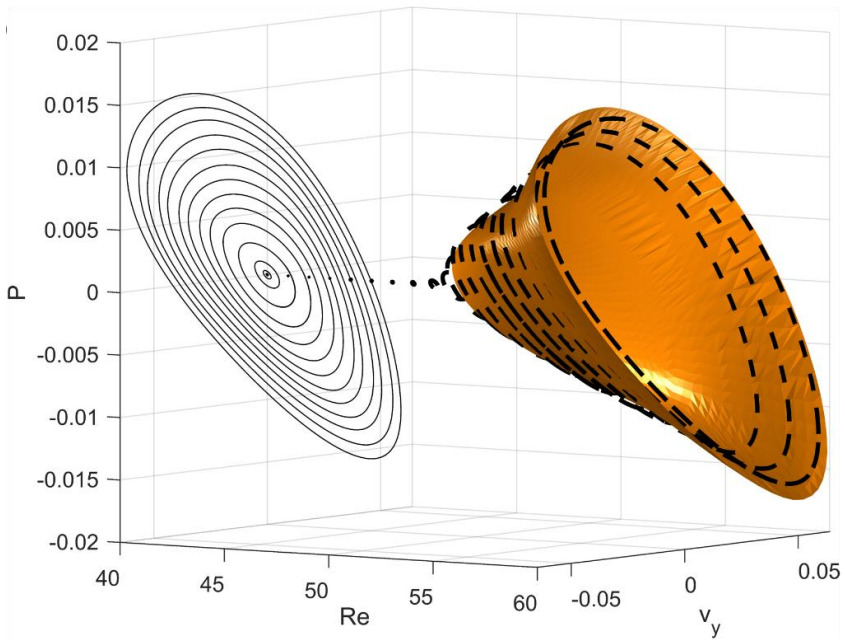
$$\dot{z}_1 = f_1(\mathbf{z}; Re)$$
$$\dot{z}_2 = f_2(\mathbf{z}; Re)$$
$$\dot{z}_3 = f_3(\mathbf{z}; Re)$$

Latent periodic solutions





Physical periodic solutions



Conti, P., Gobat, G., Fresca, S., Manzoni, A. and Frangi, A., 2022. Reduced order modeling of parametrized systems through autoencoders and SINDy approach: continuation of periodic solutions. arXiv preprint arXiv:2211.06786.

Summary

The proposed AE+SINDy approach allows to *simultaneously* achieve:

- **Dimensionality reduction**,
- **Identification** of right set of **variables**,
- **Identification** of an interpretable **dynamical model**,
- Detection of the main dynamical features of the phenomenon such as **instabilities**, **bifurcations**, etc.

Possible directions for improvement:

- Improve system identification to deal with **noisy data** and inaccurate derivatives
  → Ensemble SINDy[1], weak-formulations[2], ….
- **Normal form** identification of the observed dynamical system.

[1] Fasel, U., Kutz, J.N., Brunton, B.W. and Brunton, S.L., 2022. Ensemble-SINDy: Robust sparse model discovery in the low-data, high-noise limit, with active learning and control. *Proceedings of the Royal Society A*, *478*(2260), p.20210904.
[2] Messenger, D.A. and Bortz, D.M., 2021. Weak SINDy for partial differential equations. *Journal of Computational Physics*, *443*, p.110525.

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Thank you

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References:

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- Champion K, Lusch B, Kutz JN, Brunton SL. Data-driven discovery of coordinates and governing equations. Proceedings of the National Academy of Sciences. 2019 Nov 5;116(45):22445-51.
- Brunton SL, Proctor JL, Kutz JN. Discovering governing equations from data by sparse identification of nonlinear dynamical systems. Proceedings of the national academy of sciences. 2016 Apr 12;113(15):3932-7.