Ramanujan Machine – generating conjectures on fundamental constants

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Fundamental mathematical constants such as e and π are ubiquitous in diverse fields of science, from abstract mathematics and geometry, to physics, biology, and chemistry. Nevertheless, new mathematical formulas relating fundamental constants have been scarce for centuries and are only discovered sporadically. Such discoveries are often considered an act of mathematical ingenuity or profound intuition by great mathematicians, such as Gauss and Ramanujan. We propose systematic approaches that leverage numerical algorithms to discover mathematical formulas for fundamental constants.

Our algorithmic approach is called the '*Ramanujan Machine*'.[1][2] Over the past two years, our algorithms found dozens of well-known formulas as well as hundreds of previously unknown ones, such as continued fraction representations of π, e, Catalan's constant G, and values of the Riemann zeta function ζ. Several conjectures found by our algorithms were (in retrospect) simple to prove, whereas others remain as yet unproved. For instance:

$$e = 3 - \cfrac{1}{4 - \cfrac{2}{5 - \cfrac{3}{6 \cdots - \cfrac{n}{3+n}}}} \qquad \frac{1}{2G} \overset{?}{=} 1 - \cfrac{2}{7 - \cfrac{32}{19 - \cfrac{162}{37 \cdots - \cfrac{2n^4}{1+3n(n+1)}}}}$$

These discoveries recently unveiled a new underlying structure, the 'conserving lattice', which unifies the large variety of formulas, generalizes them to infinite families, and points to an underlying hierarchy that connects them. This structure also provides a novel approach to prove the irrationality of certain constants – a highly desired feat in many cases.

We present a few algorithms that proved useful in finding conjectures: a variant of the meet-in-the-middle algorithm, a gradient descent optimization algorithm tailored to the recurrent structure of continued fractions, and a new approach based on analysis of the greatest common divisors of recurrent formulas.[1][3][4] The latter is now run by hundreds of volunteers in the 'Ramanujan Machine' BOINC project.[5] All three algorithms are based on numerical estimates, and consequently they conjecture formulas without providing proofs or requiring prior knowledge of the underlying mathematical structure. This methodology is thus complementary to automated theorem proving.

The most recent advance in our project is the construction of a library of mathematical constants and their integer relations.[6] The vision behind this library is to make the research of these fascinating constants more accessible for the wider community. We provide automated numerical tools for finding connections between constants, for example using the PSLQ algorithm to find integer relations, toward the construction of a hypergraph relating many constants. Example relations include:

$$\frac{1}{\zeta(5) - \zeta(4) + \zeta(3) - \zeta(2) + 1} \overset{?}{=} 2 - \cfrac{2}{49 - \cfrac{1{,}536}{356 - \cfrac{78{,}732}{1{,}523 \cdots - \cfrac{n^9(n+1)}{n^5 + (n+1)^5 + (n+1)^4}}}}$$

In summary, our work supports a conceptual framework that reverses the conventional usage of sequential logic in formal proofs: computer algorithms that use numerical data to hint at previously unknown mathematical structures, leading to further mathematical research.

References:

[1] Raayoni, G., Gottlieb, S., Manor, Y. et al. Generating conjectures on fundamental constants with the Ramanujan Machine. Nature 590, 67–73 (2021)
[2] Ramanujan Machine website
[3] Razon, O., Harris, Y., Gottlieb, S., Carmon, D., David, O., & Kaminer, I. (2022). Automated Search for Conjectures on Mathematical Constants using Analysis of Integer Sequences *arXiv*:2212.09470 (2022)
[4] Ben David, N., Nimri, G., Mendlovic, U., Manor, Y., Kaminer, I. On the Connection Between Irrationality Measures and Polynomial Continued Fractions *arXiv*:2111.04468 (2021)
[5] Ramanujan Machine on BOINC website
[6] LIReC – Library of Integer Relations and Constants on Github